JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **27**, No. 3, August 2014
http://dx.doi.org/10.14403/jcms.2014.27.3.489

# THE LEBESGUE DELTA INTEGRAL

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Abstract. In this paper, we define the extension $f^* : [a, b] \to \mathbb{R}$ of a function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ for a time scale $\mathbb{T}$ and investigate the properties of the Lebesgue delta integral of $f$ on $[a, b]_{\mathbb{T}}$ by using the function $f^*$.

## 1. Introduction and preliminaries

The Lebesgue delta integral was introduced by Bohner and Guseinov in [3]. In this paper, the relationship between Lebesgue and Lebesgue delta integral is established.

Let $\mathbb{T}$ be a time scale. For every $x, y \in \mathbb{T}$ with $x < y$, we define the bounded intervals in $\mathbb{T}$ by

$$[x, y)_{\mathbb{T}} = \{t \in \mathbb{T} : x \le t < y\} \quad \text{and} \quad [x, y]_{\mathbb{T}} = \{t \in \mathbb{T} : x \le t \le y\}.$$

Now we define a countably additive measure $m$ on the set

$$\mathcal{F} = \{[x, y)_{\mathbb{T}} : x, y \in \mathbb{T}, x < y\}$$

that assigns to each interval $[x, y)_{\mathbb{T}}$ its length

$$m\Big([x, y)_{\mathbb{T}}\Big) = y - x.$$

Using $m$, we generate the outer measure $m^*$ on $\mathcal{P}([a, b]_{\mathbb{T}})$, defined for each $E \in \mathcal{P}([a, b]_{\mathbb{T}})$ as

$$m^*(E) = \begin{cases} \inf \sum_i (y_i - x_i) & \text{if} \quad b \notin E \\ +\infty & \text{if} \quad b \in E, \end{cases}$$

where the infimum is taken over all countable collection $\{[x_i, y_i)_{\mathbb{T}}\}$ of intervals such that $E \subset \cup_i [x_i, y_i)_{\mathbb{T}}$.

Received May 22, 2014; Accepted June 30, 2014.
2010 Mathematics Subject Classification: Primary 26A39; Secondary 26E70.
Key words and phrases: time scales, Lesbesgue delta integral, $\delta$-partition.
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A set $E \subset [a,b]_{\mathbb{T}}$ is $\Delta-$measurable if

$$m^*(A) = m^*(A \cap E) + m^*(A \cap ([a,b]_{\mathbb{T}} - E))$$

for each subset $A \subset [a,b]_{\mathbb{T}}$.

Defining the family

$$\mathcal{M}(m^*) = \{E \subset [a,b]_{\mathbb{T}} : E \text{ is } \Delta - \text{measurable}\},$$

the Lebesgue $\Delta-$measure, denoted by $\mu_\Delta$, is the restriction of $m^*$ to $\mathcal{M}(m^*)$.

## 2. The Lebesgue delta integral

Definition 2.1. A function $f : [a,b]_{\mathbb{T}} \to \overline{\mathbb{R}} \equiv [-\infty, \infty]$ is $\Delta-$measurable if for every $\alpha \in \mathbb{R}$, the set

$$f^{-1}([-\infty, \alpha)) = \{t \in [a,b]_{\mathbb{T}} : f(t) < \alpha\}$$

is $\Delta-$measurable.

Definition 2.2. A function $\mathcal{S} : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is simple if it only takes a finite number of different values $\alpha_1, \cdots, \alpha_n$. If $A_j = \{t \in [a,b]_{\mathbb{T}} : \mathcal{S}(t) = \alpha_j\}$, then

$$\mathcal{S} = \sum_{j=1}^{n} \alpha_j \chi_{A_j}.$$

Definition 2.3. Let $E \subset [a,b]_{\mathbb{T}}$ be a $\Delta-$measurable set and let $\mathcal{S} : [a,b]_{\mathbb{T}} \to [0,\infty)$ be a simple and $\Delta-$measurable function with

$$\mathcal{S} = \sum_{j=1}^{n} \alpha_j \chi_{A_j}.$$

The Lebesgue $\Delta-$integral of $\mathcal{S}$ on $E$ is defined by

$$(L_\Delta) \int_E \mathcal{S} = \sum_{j=1}^{n} \alpha_j \mu_\Delta(A_j \cap E).$$

Definition 2.4. Let $E \subset [a,b]_{\mathbb{T}}$ be a $\Delta-$measurable set and let $f : [a,b]_{\mathbb{T}} \to [0,\infty]$ be a $\Delta-$measurable function. The Lebesgue $\Delta-$integral of $f$ on $E$ is defined by

$$(L_\Delta) \int_E f = \sup (L_\Delta) \int_E \mathcal{S},$$

where the supremum is taken on all simple $\Delta-$measurable functions $\mathcal{S}$ such that $0 \le \mathcal{S} \le f$ on $[a,b]_{\mathbb{T}}$.

Definition 2.5. Let $E \subset [a,b]_{\mathbb{T}}$ be a $\Delta-$measurable set and let $f : [a,b]_{\mathbb{T}} \to \overline{\mathbb{R}}$ be a $\Delta-$integrable function. The function $f$ is Lebesgue $\Delta-$integrable(or $L_\Delta-$integrable) on $E$ if at least one of the elements

$$(L_\Delta)\int_E f^+ \quad \text{or} \quad (L_\Delta)\int_E f^-$$

is finite, where the positive and negative parts of $f$, $f^+$ and $f^-$ respectively, are defined as

$$f^+ = \max\{f, 0\} \quad \text{and} \quad f^- = \max\{-f, 0\}.$$

In this case, the Lebesgue $\Delta-$integral of $f$ on $E$ is defined by

$$(L_\Delta)\int_E f = (L_\Delta)\int_E f^+ - (L_\Delta)\int_E f^-.$$

Let $\{(a_k, b_k)\}_{k=1}^{\infty}$ be the sequence of all contiguous intervals of $[a,b]_{\mathbb{T}}$ in $[a,b]$.

For a function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$, define the extension $f^* : [a,b] \to \mathbb{R}$ of $f$ by

$$f^*(t) = \begin{cases} f(a_k) & \text{if} \quad t \in (a_k, b_k) \quad \text{for} \quad \text{some} \quad k \\ f(t) & \text{if} \quad t \in [a,b]_{\mathbb{T}}. \end{cases}$$

From [4, Theorem 5.1], we can easily get the following theorem.

Theorem 2.6. *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a $\Delta-$measurable function and let $f^* : [a,b] \to \mathbb{R}$ be the extension of $f$ to $[a,b]$. Then $f$ is $L_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ if and only if $f^*$ is Lebesgue integrable on $[a,b]$. In that case,*

$$(L_\Delta)\int_a^b f = (L)\int_a^b f^*.$$

Theorem 2.7. *Let $f$ be $L_\Delta-$integrable on $[a,b]_{\mathbb{T}}$. Then $f$ is $L_\Delta-$integrable on every subinterval $[c,d]_{\mathbb{T}}$ of $[a,b]_{\mathbb{T}}$.*

*Proof.* Let $f$ be a $L_\Delta-$integrable function on $[a,b]_{\mathbb{T}}$. By Theorem 2.6, $f^* : [a,b] \to \mathbb{R}$ is Lebesgue integrable on $[a,b]$. By the property of the Lebesgue integral, $f^*$ is Lebesgue integrable on every subinterval $[c,d] \subset [a,b]$. By Theorem 2.6, $f$ is $L_\Delta-$integrable on every subinterval $[c,d]_{\mathbb{T}} \subset [a,b]_{\mathbb{T}}$. □

Theorem 2.8. *Let $f$ and $g$ be $L_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and $\alpha$, $\beta$ be real numbers. Then $\alpha f + \beta g$ is $L_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and*

$$(L_\Delta)\int_a^b (\alpha f + \beta g) = \alpha (L_\Delta)\int_a^b f + \beta (L_\Delta)\int_a^b g.$$

*Proof.* Let $f$ and $g$ be $L_\Delta-$integrable on $[a,b]_{\mathbb{T}}$. By Theorem 2.6, $\alpha f^* + \beta g^*$ is Lebesgue integrable on $[a,b]$ and

$$(L)\int_a^b (\alpha f^* + \beta g^*) = \alpha\ (L)\int_a^b f^* + \beta\ (L)\int_a^b g^*.$$

By Theorem 2.6, $\alpha f + \beta g$ is $L_\Delta-$integrable on $[a,b]_\Delta$ and

$$(L_\Delta)\int_a^b (\alpha f + \beta g) = \alpha\ (L_\Delta)\int_a^b f + \beta\ (L_\Delta)\int_a^b g.$$

□

THEOREM 2.9. *Let $c \in \mathbb{T}$ with $a < c < b$. If $f$ is $L_\Delta-$integrable on each of intervals $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, then $f$ is $L_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and*

$$(L_\Delta)\int_a^b f = (L_\Delta)\int_a^c f + (L_\Delta)\int_c^b f.$$

*Proof.* If $f$ is $L_\Delta-$integrable on each of intervals $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, then $f^*$ is Lebesgue integrable in $[a,c]$ and $[c,b]$. By the property of the Lebesgue integral, $f^*$ is Lebesgue integrable on $[a,b]$ and

$$(L)\int_a^b f^* = (L)\int_a^c f^* + (L)\int_c^b f^*.$$

By Theorem 2.6, $f$ is $L_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and

$$(L_\Delta)\int_a^b f = (L_\Delta)\int_a^c f + (L_\Delta)\int_c^b f.$$

□

THEOREM 2.10. Let $\{f_n\}$ be a monotone sequence of $L_\Delta-$integrable functions on $[a,b]_{\mathbb{T}}$. Suppose that $\lim_{n\to\infty}(L_\Delta)\int_a^b f_n$ is finite. If $\lim_{n\to\infty} f_n(t) = f(t)$, then $f$ is $L_\Delta-$integrable and

$$(L_\Delta)\int_a^b f = \lim_{n\to\infty}(L_\Delta)\int_a^b f_n.$$

*Proof.* Let $\{f_n\}$ be a monotone sequence of $L_\Delta-$integrable functions on $[a,b]_{\mathbb{T}}$. Then $\{f_n^*\}$ is a monotone sequence of Lebesgue integrable functions on $[a,b]$. Since $(L_\Delta)\int_a^b f_n = (L)\int_a^b f_n^*$ for each $n$, $\lim_{n\to\infty}(L)\int_a^b f_n^*$ is finite and $\lim_{n\to\infty} f_n^*(t) = f^*(t)$ by the hypothesis.

By the property of the Lebesgue integral, $f^*$ is Lebesgue integrable on $[a,b]$ and $(L)\int_a^b f^* = \lim_{n\to\infty}(L)\int_a^b f_n^*$. By Theorem 2.6, $f$ is $L_\Delta-$inte grable on $[a,b]_{\mathbb{T}}$ and

$$(L_\Delta)\int_a^b f = \lim_{n\to\infty}(L_\Delta)\int_a^b f_n.$$

□

Recall that a bounded function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ if there exists a number $A$ such that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\Big|\sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}) - A\Big| < \epsilon$$

for every $\delta-$partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ of $[a,b]_{\mathbb{T}}$.

THEOREM 2.11. *If $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$, then $f$ is $L_\Delta-$integrable on $[a,b]_{\mathbb{T}}$. In this case, $(R_\Delta)\int_a^b f = (L_\Delta)\int_a^b f$.*

*Proof.* Suppose that $f$ is $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$. Then by [10, Thoerem 2.6], $f^*$ is Riemann integrable on $[a,b]$ and $(R_\Delta)\int_a^b f = (R)\int_a^b f^*$. Since $f^*$ is Lebesgue integrable on $[a,b]$, $f$ is $L_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and

$$(L_\Delta)\int_a^b f = (L)\int_a^b f^* = (R)\int_a^b f^* = (R_\Delta)\int_a^b f.$$

□

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